

Matter from Toric Geometry

PHILIP CANDELAS¹, EUGENE PEREVALOV² AND GOVINDAN RAJESH³

*Theory Group
Department of Physics
University of Texas
Austin, TX 78712, USA*

ABSTRACT

We present an algorithm for obtaining the matter content of effective six-dimensional theories resulting from compactification of F-theory on elliptic Calabi-Yau threefolds which are hypersurfaces in toric varieties. The algorithm allows us to read off the matter content of the theory from the polyhedron describing the Calabi-Yau manifold. This is based on the generalized Green-Schwarz anomaly cancellation condition.

¹ candelas@physics.utexas.edu

² pereval@physics.utexas.edu

³ rajesh@physics.utexas.edu

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1. Introduction

The dualities of String Theory have been the subject of extensive study during the last two years. Of particular interest to us here is the duality [1,2]

$$\text{Het}[K3 \times T^2, G] = \text{IIA}[\mathcal{M}] \quad (1.1)$$

between a $(0, 4)$ heterotic compactification on $K3 \times T^2$ with gauge group G , and a type IIA compactification on a Calabi–Yau manifold, \mathcal{M} [3-6].

In many relevant cases, the Calabi–Yau manifolds can be conveniently represented in terms of the toric data as was shown in [7-9](see also the important articles[10-12]). It was observed in [7] that it is the dual polyhedron, ∇ , which exhibits a regular structure which makes it possible, in particular, to determine the enhanced gauge symmetry given ∇ . It was noted also that in all the examples of heterotic/type II dual pairs the $K3$ and elliptic fibration structure shows itself in the existence of three- and two-dimensional reflexive subpolyhedra, respectively, inside ∇ . The three-dimensional reflexive subpolyhedron which corresponds to the generic $K3$ fiber was shown to contain the information about the part of the total gauge group (the only part in the examples considered in [7]) which has perturbative interpretation on the heterotic side. These results were extended to include non-perturbative gauge groups in [13].

The purpose of the present paper is to extend this dictionary to include the charged matter content of the low energy effective theories. The question of determining the charged matter content from geometry was addressed in Refs. [14-18], where it was pointed out that gauge groups are associated with curves of singularities and that intersecting curves of singularities lead to matter charged under both gauge groups. In this paper we use the requirement of anomaly cancellation [19] to relate the charged matter content directly to the toric data. The duality (1.1) applies most directly to Calabi–Yau manifolds that are elliptic fibrations. for these manifolds the charged matter is associated with the divisors of the base, B , of the fibration and the number and group representation of these fields is determined by the intersection numbers of the divisors.

The organisation of this paper is as follows. In §2, we list relevant results in toric geometry. In §3, we describe the relation between the charged matter content and intersection theory, and work out some simple examples. §4 is devoted to applying this technique to some of the models of Ref. [13]. §5 summarises our results.

2. Some Results in Toric Geometry

We will be dealing with elliptic Calabi-Yau threefolds which are described as hypersurfaces in toric varieties. Our notation and conventions follow those of our companion paper[13]. We shall briefly review the relevant results so that the present is self-contained. The reader, however, may care to refer to [13] for a fuller account.

In the following, Λ is a lattice of rank n and Δ , the Newton Polyhedron of the Calabi-Yau n -fold, is a reflexive polyhedron in Λ . We denote by Δ the polyhedron dual to Δ and by V the lattice dual to Λ . The real extension of V is denoted by $V_{\mathbb{R}}$. It has been shown in [20] that in order for a Calabi-Yau n -fold to be a fibration with generic fiber a Calabi-Yau $(n - k)$ -fold, it is necessary and sufficient that

(i) There is a projection operator $\Pi: \Lambda \rightarrow \Lambda_{n-k}$, where Λ_{n-k} is an $n - k$ dimensional sublattice, such that $\Pi(\Delta)$ is a reflexive polyhedron in Λ_{n-k} , or

(ii) There is a lattice plane, h , in $V_{\mathbb{R}}$ through the origin whose intersection with ∇ is an $n - k$ dimensional reflexive polyhedron, *i.e.* it is a slice of the polyhedron.

(i) and (ii) are equivalent conditions. In case (i) the polyhedron of the fiber appears as a *projection* while in case (ii) it appears as an *injection*, the projection and the injection being related by mirror symmetry. In particular, if the polyhedron of the $(n - k)$ -dimensional Calabi-Yau manifold exists as both a projection and an injection, then the intersection in ∇ is also a certain projection implying that the mirror manifold is a fibration with an $n - k$ dimensional Calabi-Yau manifold as the typical fiber. If (i) or (ii) hold there is also a way to see the base of the fibration torically[21]. The hyperplane h generates a $n - k$ dimensional sublattice of V . Denote this lattice V_{fiber} . Then the quotient lattice $V_{\text{base}} = V/V_{\text{fiber}}$ is the lattice in which the fan of the base lives. The fan itself can be constructed as follows. Let Π_B be a projection operator acting in V , of rank $\dim(V) - 2$, such that it projects h onto a point. Then $\Pi_B(V) = V_{\text{base}}$. When Π_B acts on ∇ the result is a k dimensional set of points in V_{base} which gives us the fan of the base if we draw rays through each point in the set. The pre-image of every ray in the base under Π_B determines the type of singularity (including the monodromy, if any) along the corresponding curve in the base in the way described in [13]. Thus, each ray is associated with a factor (which may be trivial) of the total gauge group.

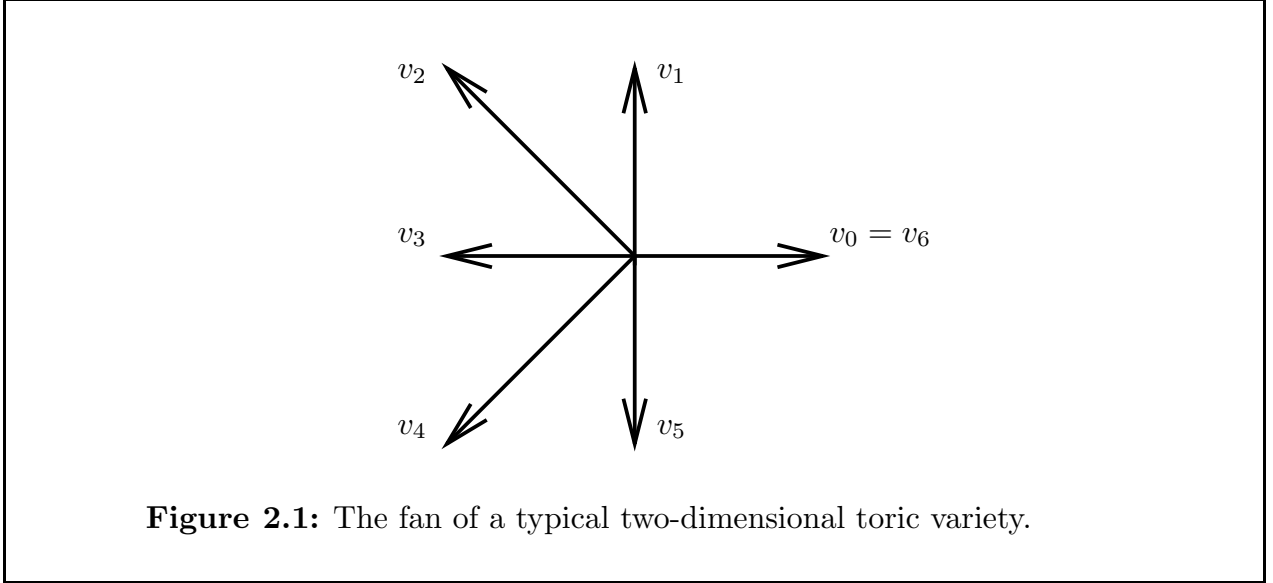
For elliptic Calabi-Yau threefolds which are hypersurfaces in toric varieties, the base of the fibration is a nonsingular two-dimensional toric variety. These are well described in §2.5 of Ref. [22] and are specified by giving a sequence of lattice points

$$v_0, v_1, \dots, v_{l-1}, v_d = v_0$$

in counterclockwise order, in $V = \mathbb{Z}^2$, such that the v_i 's are the first lattice points in each ray, and successive pairs generate the lattice (see Figure 2.1). In general, these satisfy

$$a_i v_i = v_{i-1} + v_{i+1}, \quad 1 \leq i \leq d,$$

for some integers a_i .



These two dimensional surfaces are readily classified as follows. For $d = 3$, the surface is \mathbb{P}_2 , for $d = 4$, one gets a Hirzebruch surface \mathbb{F}_n for some n . All higher values of d yield surfaces which are obtained by successive blow-ups of either \mathbb{P}_2 or \mathbb{F}_n at fixed points of the torus action.

Each v_i determines a curve $D_i \cong \mathbb{P}_1$ in the variety. The normal bundle to this embedding is the line bundle $\mathcal{O}(-a_i)$ on \mathbb{P}_1 . Successive curves meet transversally, but are otherwise disjoint:

$$(D_i \cdot D_j) = \begin{cases} 1 & \text{if } |i - j| = 1; \\ -a_i & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the canonical class of the surface is given by

$$K = - \sum_i D_i,$$

where the sum runs over all the divisors corresponding to the lattice points v_i in the fan of the base.

3. Identifying the Matter Content

In this section, we describe our approach to determining the charged matter content from the toric data. On the heterotic side, the matter content of the perturbative vacua can be found by applying the index theorem. Let $H_{1,2}$ be the background gauge groups (simple subgroups of E_8) and $k_{1,2}$ the corresponding instanton numbers (second Chern classes of the background gauge bundles on the $K3$). The contribution of each E_8 to the unbroken gauge group in six dimensions is then the commutant $G_{1,2}$ of $H_{1,2}$ respectively. The number of hypermultiplets in the representation R_a of G is then

$$N(R_a) = kT(M_a) - \dim(M_a),$$

where the adjoint of E_8 decomposes under $G \times H$ as $\mathbf{248} = \sum_a (R_a, M_a)$, and $T(M_a)$ is given by $\text{tr}(T_a^i T_a^j) = T(M_a) \delta_{ij}$, T_a^i being a generator of H in the representation M_a .

Anomaly cancellation requires $k_1 + k_2 = 24$, so it is convenient to define

$$n = k_1 - 12 = 12 - k_2$$

and take $n \geq 0$ (*i.e.*, $k_1 \geq k_2$). We find that the massless spectrum satisfies $H - V = 244$, where H and V are the numbers of massless hypermultiplets and vector multiplets, respectively. If $n \leq 8$ we can take $H_1 = H_2 = SU(2)$ and obtain $E_7 \times E_7$ gauge symmetry in six dimensions with the following matter content:

$$\frac{1}{2}(8+n)(\mathbf{56}, \mathbf{1}) + \frac{1}{2}(8-n)(\mathbf{1}, \mathbf{56}) + 62(\mathbf{1}, \mathbf{1}) .$$

If $9 \leq n \leq 12$, then k_2 cannot support an $SU(2)$ background, and the instantons in the second E_8 are necessarily small producing an unbroken E_8 . The gauge group in six dimensions is thus $E_7 \times E_8$ with matter content

$$\frac{1}{2}(8+n)(\mathbf{56}, \mathbf{1}) + (53+n)(\mathbf{1}, \mathbf{1}) .$$

Group	Charged Matter Content
$SU(2)$	$(6n + 16)\mathbf{2}$
$SU(2)_2$	$(8n + 32)\mathbf{2} + (n - 1)\mathbf{3}$
$SU(3)$	$(6n + 18)\mathbf{3}$
$SO(5)$	$(n + 1)\mathbf{5} + (4n + 16)\mathbf{4}$
G_2	$(3n + 10)\mathbf{7}$
$SU(4)$	$(n + 2)\mathbf{6} + (4n + 16)\mathbf{4}$
$SO(7)$	$(n + 3)\mathbf{7} + (2n + 8)\mathbf{8}$
$Sp(3)$	$(16 + 2n + \frac{3}{2}r)\mathbf{6} + (n + 1 - r)\mathbf{14} + \frac{1}{2}r\mathbf{14}'$
$SO(8)$	$(n + 4)(\mathbf{8}_c + \mathbf{8}_s + \mathbf{8}_v)$
$SU(5)$	$(3n + 16)\mathbf{5} + (2 + n)\mathbf{10}$
$SO(9)$	$(n + 5)\mathbf{9} + (n + 4)\mathbf{16}$
F_4	$(n + 5)\mathbf{26}$
$SU(6)$	$\frac{r}{2}\mathbf{20} + (16 + r + 2n)\mathbf{6} + (2 + n - r)\mathbf{15}$
$SO(10)$	$(n + 4)\mathbf{16} + (n + 6)\mathbf{10}$
$SO(11)$	$(\frac{n}{2} + 2)\mathbf{32} + (n + 7)\mathbf{11}$
$SO(12)$	$\frac{r}{2}\mathbf{32} + (\frac{4+n-r}{2})\mathbf{32}' + (n + 8)\mathbf{12}$
E_6	$(n + 6)\mathbf{27}$
E_7	$(\frac{n}{2} + 4)\mathbf{56}$

Table 3.1: Charged matter content of models with enhanced gauge symmetry, as a function of n .

Models with subgroups of the above can be obtained by gauge symmetry breaking via Higgs mechanism, or, equivalently, by taking the subgroups of E_8 other than $SU(2)$ as $H_{1,2}$.

Models with additional tensor multiplets, corresponding to non-perturbative heterotic vacua also exist. For these models, the massless spectrum satisfies

$$H - V = 273 - 29T \quad (3.1)$$

where T is the number of massless tensor multiplets (perturbative heterotic models have $T = 1$). We can now determine the matter content for any unbroken group. We list some of the results in Table 3.1, which appeared in Ref. [12], where it was observed that the

charged matter content is encoded on the F-theory side in the degree of vanishing of the discriminant on the locus of the corresponding singularity.

It was shown in [19] that the Green-Schwarz anomaly cancellation condition puts restrictions on the matter content of the six-dimensional gauge theories obtained on compactifying F-theory on an elliptic Calabi-Yau threefold. In particular, the amount of matter charged with respect to a group corresponding to a divisor D_i in the base of the elliptic fibration satisfies

$$\begin{aligned} \text{index}(Ad_i) - \sum_R \text{index}(R_i) n_{R_i} &= 6(K \cdot D_i) \\ y_{Ad_i} - \sum_R y_{R_i} n_{R_i} &= -3(D_i \cdot D_i), \end{aligned} \tag{3.2}$$

where in this expression n_{R_i} denotes the total number of hypermultiplets in the representation R_i of the group G_i and Ad_i denotes the adjoint representation. The $\text{index}(R_i)$ is given by $\text{trace}(T_i^a T_i^b) = \text{index}(R_i) \delta^{ab}$ and y_{R_i} is defined by decomposing $\text{tr}_{R_i} F^4 = x_{R_a} \text{tr} F^4 + y_{R_a} (\text{tr} F^2)^2$, assuming R_i has two independent fourth order invariants, and tr is the trace in a preferred representation, which is the fundamental for $SU(n)$. If R_a has only one fourth order invariant, then $x_{R_a} = 0$.

In addition, matter charged with respect to two groups satisfies

$$\sum_{R, R'} \text{index}(R_a) \text{index}(R'_b) n_{R_a R'_b} = (D_a \cdot D_b). \tag{3.3}$$

It is easy to derive the matter content for the perturbative (on the heterotic side) gauge groups listed in Table 3.1 using these formulae. Note that the value of n in the table is precisely the self-intersection of the corresponding divisor in the base. We can now also obtain the matter content for the groups which have a non-perturbative origin on the heterotic side, since the formulae above do not depend upon whether the groups are perturbative or non-perturbative from the heterotic point of view.

We illustrate this approach with a few simple examples. Consider the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string on a $K3$ manifold with one instanton shrunk to a point [23]. The corresponding F-theory dual is an elliptic fibration over F_4 , whose fan is shown in Fig. 3.1. There is an I_1^* singularity along the zero section C_0 of F_4 (which is a \mathbb{P}_1 bundle over \mathbb{P}_1), with a monodromy action on the degenerate fiber leading to $SO(9)$ gauge group as well as I_2 singularity along the divisor corresponding to \mathbb{P}_1 fiber, f , leading to an $SU(2)$ gauge group. As we know (and as is easily found from the fan of F_4) the intersection numbers

in this case are as follows:

$$C_0 \cdot C_0 = -4$$

$$f \cdot f = 0$$

$$C_0 \cdot f = 1$$

Thus, using Eq. (3.2), we find that there are $-4 + 5 = 1$ $\mathbf{9}$'s of $SO(9)$ and $6 \cdot 0 + 16 = 16$ $\mathbf{2}$'s of $SU(2)$ in the hypermultiplet spectrum. Using Eq. (3.3), we find that the charged matter content is $\frac{1}{2}(\mathbf{9}, \mathbf{2}) + \frac{23}{2}(\mathbf{1}, \mathbf{2})$, which agrees with the known result.

If we shrink two instantons at the same point of the $K3$, the $SO(9)$ of the previous example becomes $SO(10)$, and the $SU(2)$ becomes $Sp(2) \cong SO(5)$. The intersection numbers are unchanged. Applying Eq. (3.2) we obtain $-4 + 6 = 2$ $\mathbf{10}$'s of $SO(10)$ and $0 + 1 = 1$ $\mathbf{5}$'s as well as $4 \cdot 0 + 16 = 16$ $\mathbf{4}$'s of $Sp(2)$. Eq. (3.3) now fixes the matter content to be $\frac{1}{2}(\mathbf{10}, \mathbf{4}) + (\mathbf{1}, \mathbf{5}) + 11(\mathbf{1}, \mathbf{4})$, which again agrees with the known result.

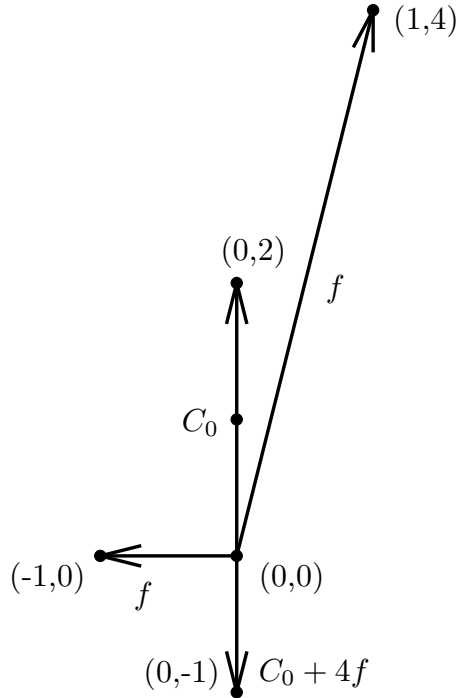


Figure 3.1: The fan of \mathbb{F}_4 . The rays of the fan are labelled by the divisors to which they correspond.

4. Examples

In this section, we apply our method to some of the examples of Ref. [13]. We have chosen to study the matter content of only those models for which the gauge content is known with certainty, *i.e.*, those for which the number of non-toric corrections is zero.

4.1. The mirrors of models with $n = 0$

(i) The mirror of the manifold with enhanced gauge group $SU(1)$

This model has gauge group $E_8^8 \times F_4^8 \times G_2^{16} \times SU(2)^{16}$ and 97 tensors. Analysis of the polyhedron reveals that all the divisors corresponding to the E_8 factors have self-intersection -12 , those corresponding to the F_4 's have self-intersection -5 , those corresponding to the G_2 's have self-intersection -3 , and those corresponding to the $SU(2)$'s all have self-intersection -2 . Furthermore, the only divisors among these that intersect each other are those corresponding to the G_2 's and the $SU(2)$'s, which intersect each other pairwise, *i.e.*, every G_2 divisor intersects exactly one divisor, which corresponds to an $SU(2)$, and vice versa. From the self-intersections, we conclude that the E_8 's and F_4 's are all matter-free, while each G_2 comes with one $\mathbf{7}$, and each $SU(2)$ comes with four $\mathbf{2}$'s. Since each G_2 intersects an $SU(2)$, we conclude that the charged matter consists of $\{\frac{1}{2}(\mathbf{7}, \mathbf{2}) + \frac{1}{2}(\mathbf{1}, \mathbf{2})\}$ from each of the 16 factors of $G_2 \times SU(2)$ for a total of 128 charged hypermultiplets. Thus, for this vacuum, $H = H_c + H_0 = 128 + 4 = 132$, $V = \dim(G) = 2672$,

$$H - V = 132 - 2672 = -2540 = 273 - 29 \times 97 = 273 - 29T,$$

so that the anomaly cancellation condition (3.1) is satisfied. We find that this is true in all the cases that we have studied, and, in particular, for the examples below. This result is no surprise, since we obtain the matter content from the Green-Schwarz anomaly cancellation condition.

(ii) The mirror of the manifold with enhanced gauge group $SU(2)$

The gauge group is $E_8^5 \times E_7^3 \times F_4^6 \times G_2^{12} \times SO(7)^2 \times SU(2)^{16}$ and, in addition, there are 81 tensors. We find that the self-intersections of the divisors are such that the E_8 's, E_7 's

and F_4 's are all matter-free. Also we find 12 pairs of mutually intersecting G_2 and $SU(2)$ divisors each of which have self-intersections -3 and -2 respectively. Here, and in future, we adopt the following shorthand for mutually intersecting divisors: we list the group factors in the form $G_1^{[a_1]} \times G_2^{[a_2]} \times \dots \times G_n^{[a_n]}$, so that the corresponding divisors have the following intersection pattern

$$(D_i \cdot D_j) = \begin{cases} 1 & \text{if } |i - j| = 1; \\ -a_i & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

where D_i is the divisor corresponding to group factor G_i . Thus we would say that there are 12 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$. We thus obtain 12 sets of

$$\{\frac{1}{2}(\mathbf{7}, \mathbf{2}) + \frac{1}{2}(\mathbf{1}, \mathbf{2})\}.$$

In addition, we get 2 factors of $SU(2)^{[-2]} \times SO(7)^{[-3]} \times SU(2)^{[-2]}$, so we find 2 sets of

$$\{\frac{1}{2}(\mathbf{2}, \mathbf{8}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{8}, \mathbf{2})\}$$

for a total of 128 charged hypermultiplets. The combination $G_2^{[-3]} \times SU(2)^{[-2]}$ occurs repeatedly in each of the examples below and in each case contributes $\{\frac{1}{2}(\mathbf{7}, \mathbf{2}) + \frac{1}{2}(\mathbf{1}, \mathbf{2})\}$.

(iii) The mirror of the manifold with enhanced gauge group $SU(3)$

The gauge group is $E_8^5 \times E_6^3 \times F_4^6 \times G_2^{10} \times SU(3)^4 \times SU(2)^{10}$ and, in addition, there are 75 tensors. The only groups that contain charged matter are 10 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ for a total of 80 charged hypermultiplets.

(iv) The mirror of the manifold with enhanced gauge group $SU(2)^2$

The group is $E_8^5 \times F_4^4 \times G_2^{10} \times SO(5)^2 \times SO(9)^2 \times SO(11)^2 \times SO(12) \times SU(2)^{12}$, and there are 67 tensors. The charged matter content comes from 10 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter charged under $SO(9)^{[-4]} \times SU(2)^{[-1]} \times SO(11)^{[-4]} \times SO(5)^{[-1]} \times SO(12)^{[-4]} \times SO(5)^{[-1]} \times SO(11)^{[-4]} \times SU(2)^{[-1]} \times SO(9)^{[-4]}$ as follows:

$$\begin{aligned} & \{\frac{1}{2}(\mathbf{9}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & + \frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{11}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & + \frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{11}, \mathbf{4}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{12}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{2}, \mathbf{9})\}
\end{aligned}$$

for a total of 216 charged hypermultiplets.

(v) *The mirror of the manifold with enhanced gauge group $SU(4)$*

The gauge group is $E_8^5 \times F_4^4 \times G_2^{10} \times SO(9)^2 \times SO(10)^3 \times SU(2)^{14}$, and there are 67 tensors. The charged matter comes from 10 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter charged under $SO(9)^{[-4]} \times SU(2)^{[-1]} \times SO(10)^{[-4]} \times SU(2)^{[-1]} \times SO(10)^{[-4]} \times SU(2)^{[-1]} \times SO(10)^{[-4]} \times SU(2)^{[-1]} \times SO(9)^{[-4]}$ as follows:

$$\begin{aligned}
& \{\frac{1}{2}(\mathbf{9}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{2}, \mathbf{10}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{10}, \mathbf{2}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{10}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& +\frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{2}, \mathbf{9})\}
\end{aligned}$$

for a total of 160 charged hypermultiplets. Note that this model can be obtained by the following Higgsing of the gauge group of the previous model:

$$SO(5) \rightarrow SU(2), SO(12) \rightarrow SO(10), \text{ and } SO(11) \rightarrow SO(10).$$

It is easy to verify that this breaking produces the charged matter content above, and precisely one additional singlet, which corresponds to the difference in the ranks of $SU(4)$ and $SU(2)^2$.

(vi) The mirror of the manifold with enhanced gauge group $SU(2) \times SU(3)$

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(6) \times SU(5)^2 \times SU(4)^2 \times SU(3)^2 \times SU(2)^{10}$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter that is charged under $SU(2)^{[-2]} \times SU(3)^{[-2]} \times SU(4)^{[-2]} \times SU(5)^{[-2]} \times SU(6)^{[-2]} \times SU(5)^{[-2]} \times SU(4)^{[-2]} \times SU(3)^{[-2]} \times SU(2)^{[-2]}$ as follows:

$$\begin{aligned}
& \{ (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{5}, \mathbf{6}, \mathbf{1}, \dots, \mathbf{1}) \\
& + 2 (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{6}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}) \}
\end{aligned}$$

for a total of 216 charged hypermultiplets.

(vii) The mirror of the manifold with enhanced gauge group $SU(5)$

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(5)^3 \times SU(4)^2 \times SU(3)^2 \times SU(2)^{10}$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter that is charged under $SU(2)^{[-2]} \times SU(3)^{[-2]} \times SU(4)^{[-2]} \times SU(5)^{[-2]} \times SU(5)^{[-2]} \times SU(5)^{[-2]} \times SU(4)^{[-2]} \times SU(3)^{[-2]} \times SU(2)^{[-2]}$ as follows:

$$\begin{aligned}
& \{ (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{5}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{5}, \mathbf{5}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}) \}
\end{aligned}$$

for a total of 204 charged hypermultiplets. It is easy to see that this matter content can be obtained from the previous one by Higgsing the $SU(6)$ to $SU(5)$, yielding exactly one extra singlet, which corresponds to the difference between the ranks of $SU(5)$ and $SU(2) \times SU(3)$.

(viii) The mirror of the manifold with enhanced gauge group $SO(10)$

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(4)^5 \times SU(3)^2 \times SU(2)^{10}$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter that is charged under $SU(2)^{[-2]} \times SU(3)^{[-2]} \times SU(4)^{[-2]} \times SU(4)^{[-2]} \times SU(4)^{[-2]} \times SU(4)^{[-2]} \times SU(4)^{[-2]} \times SU(3)^{[-2]} \times SU(2)^{[-2]}$ as follows:

$$\begin{aligned}
& \{ (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{4}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{4}, \mathbf{4}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}) \}
\end{aligned}$$

for a total of 176 charged hypermultiplets. Again, one finds that this matter content can be obtained from the previous one by Higgsing the $SU(5)$'s to $SU(4)$'s, yielding exactly one extra singlet, corresponding to the difference between the ranks of $SO(10)$ and $SU(5)$.

(ix) The mirror of the manifold with enhanced gauge group E_6

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(3)^7 \times SU(2)^{10}$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter that is charged under $SU(2)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(3)^{[-2]} \times SU(2)^{[-2]}$ as follows:

$$\begin{aligned}
& \{ (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}) \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \dots, \mathbf{1}) \\
& \vdots \\
& + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}) \}
\end{aligned}$$

for a total of 140 charged hypermultiplets. Again, one finds that this matter content can be obtained from the previous one by Higgsing the $SU(4)$'s to $SU(3)$'s, yielding exactly one extra singlet, which corresponds to the difference between the ranks of E_6 and $SO(10)$.

(x) *The mirror of the manifold with enhanced gauge group E_7*

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(2)^{17}$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ plus matter that is charged under $SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]} \times SU(2)^{[-2]}$ as follows:

$$\begin{aligned} & \{2(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & + (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{1}) \\ & \vdots \\ & + 2(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}) \} \end{aligned}$$

for a total of 104 charged hypermultiplets. Again, one finds that this matter content can be obtained from the previous one by Higgsing the $SU(3)$'s to $SU(2)$'s, yielding exactly one extra singlet, which corresponds to the difference between the ranks of E_7 and E_6 .

(xi) *The mirror of the manifold with enhanced gauge group E_8*

The gauge group is $E_8^5 \times F_4^4 \times G_2^8 \times SU(2)^8$, and there are 61 tensors. The charged matter comes from 8 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$ for a total of 64 charged hypermultiplets. Once again, this matter content can be obtained from the previous one by Higgsing the $SU(2)$'s away, yielding exactly 13 extra singlets, which corresponds to the difference between the ranks of E_8 and E_7 plus 12 extra tensor multiplets which appear in the original spectrum when the E_8 is unhiggsed.

4.2. *The mirrors of models with higher values of n .*

It is straightforward to obtain the matter content of the mirrors of models with higher values of n . We present below the matter content of the model with $n = 4, 6$ and 12 . The mirror of the model with $n = 12$ was also studied by Aspinwall and Gross [24].

(i) The mirror of the model with $n = 4$

The gauge group is $E_8^9 \times F_4^9 \times G_2^{18} \times SU(2)^{18}$, and there are 107 tensors. The charged matter comes from 18 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$. In addition, one of the divisors corresponding to a factor of F_4 has self-intersection -4 (all the others have self-intersection -5), so that we get additional charged matter transforming in the **26** of F_4 , for a total of 170 charged hypermultiplets. Note that the **26** of F_4 contains 2 zero weight vectors, so that while there are only 6 neutral hypermultiplets in the theory, we get two more upon going to the Coulomb branch. This is consistent with the fact that the Hodge numbers are $(271, 7)$.

(ii) The mirror of the model with $n = 6$

The gauge group is $E_8^{11} \times F_4^{10} \times G_2^{21} \times SU(2)^{22}$, and there are 127 tensors. The charged matter comes from 20 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$. In this case we also have one factor of $SU(2)^{[-2]} \times G_2^{[-2]} \times SU(2)^{[-2]}$, which yields

$$\left\{ \frac{1}{2}(\mathbf{2}, \mathbf{7}, \mathbf{1}) + \frac{1}{2}(\mathbf{2}, \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{7}, \mathbf{2}) + \frac{1}{2}(\mathbf{1}, \mathbf{1}, \mathbf{2}) + 2(\mathbf{1}, \mathbf{7}, \mathbf{1}) \right\}$$

for a total of 190 charged hypermultiplets. Note that the **7** of G_2 contains a zero weight vector, so that while there are only 8 neutral hypermultiplets in the theory, we get two more upon going to the Coulomb branch. This is consistent with the fact that the Hodge numbers are $(321, 9)$.

(iii) The mirror of the model with $n = 12$

The gauge group is $E_8^{17} \times F_4^{16} \times G_2^{32} \times SU(2)^{32}$, and there are 193 tensors. The charged matter comes from 32 factors of $G_2^{[-3]} \times SU(2)^{[-2]}$, each of which contribute

$$\left\{ \frac{1}{2}(\mathbf{7}, \mathbf{2}) + \frac{1}{2}(\mathbf{1}, \mathbf{2}) \right\},$$

for a total of 256 charged hypermultiplets.

5. Discussion

In this paper, we have shown how toric geometry encodes the matter content of F-theory compactifications on elliptic Calabi-Yau threefolds. The Green-Schwarz anomaly cancellation condition in six dimensions relates the matter content of gauge theories to the geometric data [19]. We find that this relation takes on a very simple form in terms of the toric data, making it possible to read off the matter content from the polyhedron describing the Calabi-Yau threefold. Thus, in this respect, we have almost completed the dictionary relating geometry and physics. In particular, given any elliptic Calabi-Yau threefold that has a toric description in terms of a reflexive polyhedron, we can read off the massless spectrum of the resulting six-dimensional theory from the polyhedron by using the methods described in this paper and Ref. [13]. All the elliptic Calabi-Yau threefolds that are hypersurfaces in toric varieties and can be obtained from a single weight system have already been constructed [20]. Combined with the results of this paper and Ref. [13], this could be interpreted as meaning that a large class of $N = 1$ vacua in six dimensions can be constructed using toric methods.

There is a point which is worth mentioning here. For every elliptic Calabi-Yau threefold described by a reflexive polyhedron, the base is always a non-singular two-dimensional toric variety. It is easy to see that all the divisors corresponding to distinct rays in the fan of the base describe genus zero curves. Now, in Ref. [25], it was shown that a genus g curve of A_{n-1} singularities in an elliptic Calabi-Yau threefold leads to an enhanced $SU(n)$ gauge symmetry with g adjoint hypermultiplets. This means that if a manifold with such a curve of singularities can be described as a hypersurface in a toric variety, then the divisor in the base obtained by projecting out the divisor corresponding to the singularity cannot be seen as a ray in the base, and our methods of determining the gauge and matter content of the theory break down. However, this does not imply that one cannot describe vacua with adjoint hypermultiplets using toric methods. For instance, the model with $n = 2$ and enhanced gauge group $SU(2)_c$ (in the notation of Ref. [7]), obtained using toric geometry, has charged matter content consisting of 1 $\mathbf{3}$ and 48 $\mathbf{2}$'s, even though the genus of this curve of singularities is zero¹.

There is an interesting property of the models described in §4. Consider the Higgsing of the E -series

$$E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow E_5 = SO(10) \rightarrow E_4 = SU(5) \rightarrow E_3 = SU(2) \times SU(3).$$

¹ We thank P. Berglund for useful discussions on this point.

For all n , models with these enhanced gauge groups have mirrors whose gauge contents form a Higgsing sequence in the opposite direction, for example, the mirror of the model with enhanced gauge group E_7 has a gauge content which can be Higgsed to yield the mirror of the model with gauge group E_8 and the same value of n . The actual Higgsing pattern is described in §5, for models with $n = 0$. We have found that this Higgsing pattern persists for all values of n , even though the actual gauge groups are different. In all these cases, the number of tensors in the mirror models is constant. Continuing the Higgsing sequence further,

$$E_3 = SU(2) \times SU(3) \rightarrow E_2 = SU(2)^2 \rightarrow E_1 = SU(2) \rightarrow E_0 = SU(1),$$

we find that the number of tensors in the mirror models increases each time, and the mirrors are not related by Higgsing in any way. This relation between Higgsing and unhiggsing of the models and their mirrors is very intriguing. It would be interesting to understand the physical meaning of this phenomenon and its relation, if any, to the duality in three dimensional $N = 4$ theories proposed by Intriligator and Seiberg [26,27].

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